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RELATIONSHIPS BETWEEN λ-NUCLEARITY AND PSEUDO-μ-NUCLEARITY

BY

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ABSTRACT. It is shown that for any Köthe space λ, λ-nuclearity coincides with pseudo-$\lambda \cdot \lambda^{\times}$-nuclearity. More particular results, including a Grothendieck-Pietsch criterion for λ-nuclearity of sequence spaces, are given for Köthe spaces which are regular.

There has been considerable activity in recent years centered upon the generalizations of the concept of nuclearity which are obtained by exchanging the sequence space l_1 for other sequence spaces λ. In one direction, Pietsch and Persson make use of the l_p-spaces [11] and in another direction, Brudovskiĭ uses the space (s) of rapidly decreasing sequences [2], which was followed by Ramanujan with the spaces $\Lambda_\infty(\alpha)$ [13].

In their Memoir [6], Dubinsky and Ramanujan attained a unification of these theories and two useful concepts emerged: λ-nuclearity and pseudo-λ-nuclearity. It is the purpose of this paper to show that for a large class of sequence spaces λ, there is a natural equivalence of λ-nuclearity with an appropriate pseudo-μ-nuclearity.

In the first section, we give all relevant.definitions and notation. §2 contains the central equivalence theorem and demonstrates its relationship to the preceding work of Dubinsky and Ramanujan in [6], and of the author in [14]. §3 contains applications to Köthe spaces with regular bases, and a very general Grothendieck-Pietsch criterion is obtained.

In the fourth section, various inclusion theorems are given. In particular, it is shown that for any nuclear Köthe space λ, any space $\Lambda(P)$ which is λ-nuclear must be (s)-nuclear. §5 contains examples which mark out the boundaries of the theory for Köthe spaces.

1. **Introduction.** For general terminology on locally convex spaces we refer to [7]. For any pair of sequences a and b, we shall write $a \cdot b$ for the sequence whose nth term is $a_n b_n$. We say that a is *dominated by* b, denoted

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$a \prec b$, if there exists $M > 0$ such that $|a_n| \leqslant M \cdot |b_n|$ for all $n \in \mathrm{N}$. (N denotes the collection of positive integers.) A *sequence space* λ is a vector space of sequences which contains $\varphi = \{(t_n): t_n = 0$ except for finitely many $n \in \mathrm{N}\}$. A sequence space λ is said to be *solid*, or *normal*, if whenever $b \in \lambda$ and $a \prec b$ then $a \in \lambda$. The Köthe *dual*, $\lambda^\times$, of a sequence space λ is defined by $\lambda^\times = \{(t_n): \Sigma_{n=1}^\infty |a_n||t_n| < +\infty$ for all $a \in \lambda\}$. λ is said to be perfect if $\lambda = \lambda^{\times\times}$. We denote by l_1 the space of absolutely summable scalar sequences, and by l_∞ the space of bounded sequences. c_0 denotes the collection of sequences which converge to 0. For a sequence a and a sequence space λ, $a \cdot \lambda = \{a \cdot b: b \in \lambda\}$.

A collection of sequences P is called a *Köthe set* if $a \geqslant 0$ for all $a \in P$, if for each $a, b \in P$ there exists $c \in P$ such that $a \prec c$ and $b \prec c$, and if for each $n \in \mathrm{N}$ there exists $a \in P$ such that $a_n > 0$. We define the locally convex space $\Lambda = \Lambda(P)$ by $\Lambda(P) = \{t: \|t\|_a = \Sigma\, |t_n| a_n < +\infty$ for all $a \in P\}$. and give $\Lambda(P)$ the topology generated by the seminorms $\|\ \|_a$. It is easy to check that $\Lambda(P)$ is a complete locally convex space.

In particular, if P is countable and if for each $a \in P$, $a_n > 0$ for all n, we call $\Lambda(P)$ a *Köthe space.* It is easy to check that we can assume that $P = \{a^k\}_{k=1}^\infty$ where $0 < a_n^k \leqslant a_n^{k+1}$ for all k, n. We then write $\Lambda(P) = \bigcap_{k=1}^\infty (1/a^k) \cdot l_1$. It is well known that $\Lambda(P)^\times = \bigcup_k a^k \cdot l_\infty$, and of course each Köthe space is a Fréchet space, with the topology induced by the norms $\|\ \|_k = \|\ \|_{a^k}$, $k = 1, 2, \cdots$. The space $\bigcap (1/a^k) \cdot l_1$ is a *Schwartz space* if and only if for all k there exists m such that $a^k/a^m \in c_0$, and is a *nuclear space* if and only if for each k there exists m such that $a^k/a^m \in l_1$ ([3], and [10, 6.1.5]).

Let α be a sequence such that $0 < \alpha_n \leqslant \alpha_{n+1}$ for all n. We define the *power series space of infinite type*, $\Lambda_\infty(\alpha)$, to be the Köthe space generated by $a_n^k = k^{\alpha_n}$. The *power series spaces of finite type*, $\Lambda_p(\alpha)$, are the Köthe spaces generated by $a_n^k = ((k/(k+1)) \cdot p)^{\alpha_n}$, $p > 0$. Of particular importance is the space $(s) = \Lambda_\infty(\alpha)$ with $\alpha_n = \log(n+1)$. (s) is called the *space of rapidly decreasing sequences.*

The *nth coordinate sequence* e^n is the sequence whose nth term is 1 and whose other terms are 0. Then (e^n) is an absolute basis for every sequence space $\Lambda(P)$, in the sense that if $t \in \Lambda(P)$ then $t = \Sigma_{n=1}^\infty t_n e^n$, and $\Sigma_{n=1}^\infty |t_n|\, \|e^n\|_a < +\infty$ for all $a \in P$. The sequence e has each coordinate equal to 1.

We shall say that a Köthe space λ is *regular* if $\lambda = \bigcap (1/a^k) \cdot l_1$, where for each k and n, $a_n^k/a_n^{k+1} \geqslant a_{n+1}^k/a_{n+1}^{k+1}$; i.e., a^k/a^{k+1} is decreasing. We then say that the matrix (a_n^k) is *regular.*

A Köthe space λ will be called a (d_1) space if $\lambda = \bigcap (1/a^k) \cdot l_1$, where there exists k such that for all m there exists p such that $(a^m)^2/(a^k \cdot a^p) \in l_\infty \cdot \lambda$ will be called a (D_1) space if $\lambda = \bigcap (1/a^k) \cdot l_1$ where the sequences a^k are such that (1) $a_n^1 = 1$ for all n, and (2) for each m there exists p such that $(a^m)^2 \leqslant a^p$.

A Köthe space λ will be called a (d_2) space if $\lambda = \bigcap (1/a^k) \cdot l_1$, where for each k there exists m such that for each p, $(a^k \cdot a^p)/(a^m)^2 \in l_\infty \cdot \lambda$ will be called a (D_2) space if $\lambda = \bigcap (1/a^k) \cdot l$ where the sequences a^k are such that (1) for each n, $\lim_{k\to\infty} a_n^k = 1$, and (2) for each k there exists m such that for each p, $a^k \cdot a^p \leqslant (a^m)^2$.

The notions of regularity and (d_1) and (d_2) were first introduced by Dragilev in [5], and Bessaga defined (D_1) and (D_2) spaces in [1]. Note that here we follow Zaharjuta [16] and do not require regularity in the (d_i) or (D_i) conditions. Bessaga denotes regular matrices (a^k) by (D_0). Every (d_i) space is a diagonal transform of a (D_i) space [5].

Let λ be a sequence space, and let E and F be normed linear spaces (n.l.s.). We say that a linear map $T: E \longrightarrow F$ is λ-nuclear if there exists $\xi \in \lambda$, (a^n) bounded in E' and $(y^n) \subseteq F$ such that $(\langle y^n, b\rangle) \in \lambda^\times$ for all $b \in F_j'$ and such that $Tx = \Sigma_{n=1}^\infty \xi_n \langle x, a^n\rangle y^n$, for all $x \in E$. We denote by $N_\lambda(E, F)$ the collection of λ-nuclear maps from E to F.

A linear map T from E to F will be called a *pseudo-λ-nuclear* map if there exists $\xi \in \lambda$, (y^n) bounded in F, and (a^n) bounded in E' such that $Tx = \Sigma\, \xi_n \langle x, a^n\rangle y^n$. We denote by $\widetilde{N}_\lambda(E, F)$ the collection of pseudo-λ-nuclear maps of E into F.

λ-nuclear maps have been studied by Pietsch and Persson [11] in the case when $\lambda = l_p$, $1 \leqslant p < \infty$. Several authors have studied pseudo-λ-nuclear mappings under the name of λ-nuclear mappings (cf. [2], [8], [13]). The term pseudo-λ-nuclearity was introduced in the Memoir of Dubinsky and Ramanujan, who also proved that for all spaces $\Lambda_\infty(\alpha)$ and for all n.l.s. E and F, $N_{\Lambda_\infty(\alpha)}(E, F) = \widetilde{N}_{\Lambda_\infty(\alpha)}(E, F)$ [6, Theorem 1.1]. In [15] Terzioğlu investigated pseudo-λ-nuclearity for certain spaces $\lambda = \lambda(P)$. In [14], the equality of $N_{\Lambda_1(\alpha)}(E, F)$ and $\widetilde{N}_{\Lambda_1(\alpha)}(E, F)$ was established for all n.l.s. E and F and for all nuclear spaces $\Lambda_1(\alpha)$.

Following [6] we say that a l.c.s. E is *(pseudo-λ-nuclear)* if for all barreled 0-neighborhoods U in E there exists a barreled 0-neighborhood V in E which is absorbed by U and such that the canonical map $E(V, U)$ is (pseudo)-λ-nuclear. We shall denote the collection of λ-nuclear spaces by N_λ and the collection of pseudo-λ-nuclear spaces by $\widetilde{N}_\lambda$.

For a sequence ξ and sequence spaces λ and μ such that $\xi \cdot \lambda \subseteq \mu$, we define the *diagonal map from* λ *to* μ *with diagonal* ξ by $T_\xi(\eta) = \xi \cdot \eta$ for all $\eta \in \lambda$. The collection of diagonal maps from λ to μ will be denoted by

$D(\lambda, \mu)$. If λ and μ are perfect then $D(\lambda, \mu) = (\lambda \cdot \mu^\times)^\times$ [3, Proposition 1.2]. For a continuous linear map T from the n.l.s. E to the n.l.s. F we define the *nth approximation numbers* of T by $\alpha_n(T) = \inf\{\|T - A\|: A$ is a finite rank map of E to F with $\dim A(E) \leqslant n\}$, for $n = 0, 1, 2, \cdots$, T is said to be of *type* λ if $(\alpha_{n-1}(T))_{n=1}^\infty \in \lambda$ (cf. [10, Chapter 8]). The *nth diameter* of T is defined by $\delta_n(T) = \inf\{\delta > 0: T(U) \subseteq \delta V + L$, L a subspace of F, dimension $L \leqslant n$, and U and V the unit balls of E and $F\}$. Then Proposition (9.1.6) of [10] states that, for all $n = 0, 1, 2, \cdots$, $\delta_n(T) \leqslant \alpha_n(T) \leqslant (n+1)\delta_n(T)$.

It is well known that if $T_\xi: l_1 \longrightarrow l_1$ is a diagonal map with diagonal $\xi \in c_0$, then, for all $n = 1, 2, \cdots, \alpha_{n-1}(T_\xi) = \delta_{n-1}(T_\xi) = \hat{\xi}_n$ where $\hat{\xi}$ is the *decreasing rearrangement of* ξ as defined in [6, p. 8].

If $A: l_1 \longrightarrow l_1$ is a linear map and if $\langle Ae^i, e^j\rangle = a_{ij}$ for all i, j we say that A is represented by (a_{ij}) and write $A \sim (a_{ij})$.

Given two sequence spaces λ_1 and λ_2 we define the sequence space $\lambda_1 \vee \lambda_2$ by $\lambda_1 \vee \lambda_2 = \{t: (t_{2n}) \in \lambda_1$ and $(t_{2n-1}) \in \lambda_2\}$. Clearly $\lambda_1 \vee \lambda_2$ is algebraically isomorphic to the cartesian product $\lambda_1 \times \lambda_2$

2. Fundamental results.

(2.1) THEOREM. *Let* $\lambda = \bigcap (1/a^k) \cdot l_1, \mu = \lambda \cdot \lambda^\times$. *Then, for all n.l.s. E and F,* $N_\lambda(E, F) = \widetilde{N}_\mu(E, F)$, *and* $N_\lambda = \widetilde{N}_\mu$.

PROOF. Let $Tx = \sum \xi_n \langle x, a_n\rangle y_n$, with $\xi \in \lambda$, (a_n) bounded in E' and $(y_n) \in \lambda^\times(F)$. Then by Lemma 1.3 of [6], $\{(\langle y_n, b\rangle): \|b\| = 1, b \in F'\}$ is bounded in $T_s(\lambda, \lambda^\times)$, and hence there exists a^k such that $\|y_n\| \leqslant a_n^k$ for all n. Then

$$Tx = \sum \xi_n \|y_n\| \langle x, a_n\rangle y_n / \|y_n\|,$$

and this is a pseudo-μ-nuclear representation for F. Conversely if $Tx = \sum \xi_n \langle x, a_n\rangle y_n$, with $\xi = \xi^1 \cdot a^k \in \lambda \cdot \lambda^\times$, and $(y_n) \in l_\infty(F)$, then $Tx = \sum \xi_n^1 \langle x, a_n\rangle a_n^k y_n$, and $(a_n^k y_n) \in \lambda^\times(F)$. □

(2.2) COROLLARY. *If* $\lambda = \bigcap (1/a^k) \cdot l_1$ *is nuclear, then for all n.l.s. E and F,* $N_\lambda(E, F) = N_{\lambda^\times}(E, F)$.

PROOF. This follows from the fact that if $(y_n) \subseteq F$ and if $(\langle y_n, b\rangle)_n \in \lambda$ for all $b \in F'$, then there exist $b \in \lambda$ such that $\|y_n\| \leqslant b_n$ for all n (cf. [9, p. 270]). □

(2.3) REMARK. Observe that if $\lambda = \Lambda_\infty(\alpha)$ is nuclear then $\lambda \cdot \lambda^\times = \lambda$. Hence Theorem (2.1) contains Theorem (1.1) of [6]. Also if $\lambda = \Lambda_1(\alpha)$ is nuclear, then $\lambda \cdot \lambda^\times = \lambda^\times$, so Theorem (2.1) contains Theorem (2.4) of [14]. The following propositions demonstrate the limits of the equalities $\lambda \cdot \lambda^\times = \lambda$ and $\lambda \cdot \lambda^\times = \lambda^\times$.

(2.4) PROPOSITION. *Let* $\lambda = \bigcap (1/a^k) \cdot l_1$ *be nuclear.* $\lambda \cdot \lambda^\times = \lambda$ *if and only if* λ *is* (D_1).

PROOF. Suppose $\lambda \cdot \lambda^\times \supseteq \lambda$. Then $l_\infty \subseteq \lambda^\times$ so $\exists k_0 \ni e \prec a^{k_0}$. Moreover, if $\lambda \cdot \lambda^\times \subseteq \lambda$, then $\lambda^\times \subseteq D(\lambda, \lambda)$, so for all k, $a^k \in D(\lambda, \lambda)$. Hence for all k and m, there exists j such that $(a^k \cdot a^m)/a^j \in l_\infty$, or $a^k \cdot a^m \prec a^j$. Thus there exists a (D_1) matrix (b^k) with $\lambda = \bigcap (1/b^k) \cdot l_1$.

Conversely, if $\lambda = \bigcap (1/b^k) \cdot l_1$, with (b^k) a (D_1) matrix, then $e \in \lambda^\times$, so $\lambda \cdot \lambda^\times \supseteq \lambda$. Also, given b^k then for all m there exist j such that $b^k \cdot b^m \prec b^j$. Thus for all $x \in \lambda, b^k \cdot x \cdot b^m \prec x \cdot b^j \in l_1$, so $\lambda \cdot \lambda^\times \subseteq \lambda$. □

(2.5) PROPOSITION. *Let* $\lambda = \bigcap (1/a^k) \cdot l_1$ *be nuclear. Then* $\lambda \cdot \lambda^\times = \lambda^\times$ *if and only if the* λ *is* (D_2).

PROOF. We show that $\lambda \cdot \lambda^\times = \lambda^\times$ if and only if (i) for all k there exists m such that, for all j, $a^j \cdot a^k < a^m$, and (ii) for all m, there exists j such that $a^m/(a^j)^2 \in l_\infty$.

In fact we have $\lambda \cdot \lambda^\times \supseteq \lambda^\times$ if and only if $\bigcup_a^m \cdot \lambda \supseteq \bigcup_a^k \cdot l_\infty$, if and only if for all k there exists m with $a^k/a^m \in \lambda$, so that for all k there exists m with $(a^k/a^m) \cdot a^j \in l_\infty$ for all j. Thus $\lambda \cdot \lambda^\times \supseteq \lambda^\times$ if and only if (i) holds.

Now $\lambda \cdot \lambda^\times \subseteq \lambda^\times$ if and only if $\lambda \subseteq D(\lambda, \lambda)$, which is equivalent to the statement that $\lambda \cdot \lambda \subseteq \lambda$. Hence λ is an algebra, and we will apply the Closed Graph Theorem to conclude that the mapping $(x, y) \longrightarrow x \cdot y$ of $\lambda \times \lambda \longrightarrow \lambda$ is continuous. Consider sequences (x^i) and (y^i) in λ such that $x^i \longrightarrow x$ in λ and $y^i \longrightarrow y$ in λ. Then for all k,

$$\|(x^i \cdot y^i - x \cdot y) \cdot a^k\|_{l_1} \leq \|x^i \cdot (y^i - y)a^k\|_{l_1} + \|y \cdot (x^i - x) \cdot a^k\|_{l_1}.$$

But given y and a^k, there exist j with $a^k \cdot y \prec a^j$. Hence there exists $M > 0$ such that, for all i, $\|y \cdot (x^i - x) \cdot a^k\|_{l_1} \leq M\|(x^i - x) \cdot a^j\|_{l_1} \longrightarrow 0$, as $i \longrightarrow \infty$. Now we know that (x^i) is bounded in λ, so that by Proposition (5) of [9] there exists $y^0 \in \lambda$ with $x^i \leq y^0$ for all i. Hence there exists j' and $M' > 0$ such that

$$\|x^i \cdot (y^i - y) \cdot a^k\|_{l_1} \leq \|y^0 \cdot (y^i - y) \cdot a^k\|_{l_1} \leq M'\|(y^i - y) \cdot a^{j'}\|_{l_1},$$

which goes to 0 as i approaches ∞. Thus $x^i \cdot y^i \longrightarrow x \cdot y$, so multiplication in λ is continuous. Hence for all m there exists k such that $a^m \prec (a^k)^2$, and (ii) holds.

Conversely, given (ii) and x and y in λ we see that for all m, $x \cdot y \cdot a^m = x \cdot a^j \cdot y \cdot a^j \cdot (a^m/(a^j)^2) \in l_1 \cdot l_1 \cdot l_\infty \subseteq l_1$, so that $x \cdot y \in \lambda$. Thus $\lambda \cdot \lambda \subseteq \lambda$, so we have shown that (ii) $\Longleftrightarrow \lambda \cdot \lambda^\times \subseteq \lambda^\times$.

We must show next that conditions (i) and (ii) are equivalent to the statement that λ is (D_2). If λ is (D_2), it is straightforward to check that (i) and (ii) hold. In the other direction, we may assume without loss of generality that $\lambda = \bigcap (1/a^k) \cdot l_1$, where for all m there exists j such that $a^m/(a^j)^2 \leqslant e$. Using (i) and the nuclearity of λ we see that for all k there exists n_k such that if $n \geqslant n_k$, then $a_n^k < 1/2$, with the n_k's chosen so that $n_k < n_{k+1}$. Now define $b_n^k = 1$ if $n \leqslant n_k$ and $b_n^k = a_n^k$ if $n > n_k$. Then $\lambda = \bigcap (1/b^k) \cdot l_1$, so λ is (D_2). □

(2.6) REMARK. Theorem (2.1) provides a method of reducing questions involving λ-nuclearity to questions involving pseudo-μ-nuclearity. This is particularly useful for obtaining inclusion theorems such as Theorem (3.3) and Theorem (4.2) of the next sections. However it is of somewhat limited value in going from pseudo-μ-nuclearity to λ-nuclearity. The natural problem which arises is this: What Köthe spaces μ arise in the solutions of the equalities $\lambda \cdot \lambda^\times = \mu$ or $\lambda \cdot \lambda^\times = \mu^\times$? The former implies that $\mu \cdot \mu^\times = \lambda \cdot \lambda^\times \cdot D(\lambda \cdot \lambda) = \lambda \cdot \lambda^\times = \mu$, so that μ must be a (D_1) space by Proposition 2.4. Similarly the latter equality implies that $\mu \cdot \mu^\times = \mu^\times$, so that μ must be a (D_2) space by Proposition (2.5).

Hence, for the spaces $\Lambda_p(\alpha)$, $0 < p < \infty$, there exists no space λ such that $\lambda \cdot \lambda^\times = \Lambda_p(\alpha)$. Thus it is an open question whether there exists a Köthe space λ for which $\widetilde{N}_{\Lambda_p(\alpha)} = N_\lambda$.

It should be observed that $\lambda \cdot \lambda^\times$ is always a normal space but is not in general perfect. The existence of such a space λ is a deep result and appears in [4].

3. **Spaces with regular bases.** In this section, we consider Köthe spaces λ which are regular, and develop the most general version of the Grothendieck-Pietsch criterion for λ-nuclearity of spaces $\Lambda(P)$.

(3.1) LEMMA. *If* $\lambda = \bigcap (1/a^k) \cdot l_1$ *is regular and nuclear and if* $t \in \lambda \cdot \lambda^\times$, *then* $(\Sigma_{i=n}^\infty t_i)_n \in \lambda \cdot \lambda^\times$.

PROOF. Let $S: l_1 \longrightarrow c_0$ be the map given by $S(t) = (\Sigma_{i=n}^\infty t_i)_n$. We must show that $S(\lambda \cdot \lambda^\times) \subseteq \lambda \cdot \lambda^\times$. For this it suffices to show that $S(a^k\lambda) \subseteq a^k\lambda$ for all $k = 1, 2, \cdots$, which is equivalent to the statement that for all k, $T_{(a^k)^{-1}} \circ S \circ T_{a^k}(\lambda) \subseteq \lambda$. Since the map in question is a matrix map, we have only to show that, for each $m \geqslant k$, there exists j such that

$$\sup_n \frac{\|T_{(a^k)^{-1}} \circ S \circ T_{a^k}(e^n)\|_m}{\|e^n\|_j} < +\infty .$$

But $T_{(a^k)^{-1}} \circ S \circ T_{a^k}(e^n) = \Sigma_{i=1}^n (a_n^k/a_i^k)e^i$, for $n \geqslant 1$, so we need

$$\sup_n \sup_{1 \leqslant i \leqslant n} \frac{a_n^k}{a_i^k} \cdot \frac{a_i^m}{a_n^j} < +\infty.$$

But if $j = m$, this becomes $\sup_n a_n^k/a_n^m \cdot a_n^m/a_n^k = 1$, by regularity of the matrix (a_n^k).

(3.2) LEMMA. *If* λ *is nuclear,* $\lambda \cdot \lambda^\times \subseteq \bigcap_{p>0} l_p$.

PROOF. Let $u = x \cdot a^k \in \lambda \cdot \lambda^\times$. By nuclearity of λ there exists m such that $a^k/a^m \in l_1$; and, by induction, for each $p > 0$ there exists $m = m(k, p)$ such that $a^k/a^m \in l_p$. Hence, for each p, $u = x \cdot a^m \cdot a^k/a^m \in l_1 \cdot l_p \subseteq l_p$. □

(3.3) THEOREM. *If* λ *is a regular nuclear Köthe space then* $N_\lambda \subseteq N_{(s)}$.

PROOF. Let $t \in \lambda \cdot \lambda^\times$. By Lemma (3.1), we see that $(\Sigma_{i=n}^\infty t_i)_n \in \lambda \cdot \lambda^\times$; so, by Lemma (3.2), $\xi = (\Sigma_{i=n}^\infty |t_i|)_n \in \bigcap_{p>0} l_p$. But ξ is decreasing and thus belongs to (s) [10, 8.5.5, Lemma 2]. Since $t \prec \xi$ and (s) is normal, $t \in (s)$. Thus $\lambda \cdot \lambda^\times \subseteq (s)$, and we immediately obtain the inclusion relation $N_\lambda \subseteq N_{(s)}$ from Theorem (2.1). □

(3.4) REMARK. Theorem (2.2) of [16] states that if $\lambda[T_b(\lambda^\times)]' = \lambda^\times$ and if $\Lambda(P) \in N_\lambda$ then $\Lambda(P) \in N_{l_1}$. Theorem (3.3) above is a strengthening of that result for nuclear regular Kothe spaces λ by showing that every λ-nuclear space is (s)-nuclear. In the next section we will give a version of this result valid for any Köthe space λ which is nuclear, not necessarily regular.

(3.5) THEOREM. *Let* $\lambda = \bigcap (1/a^k) \cdot l_1$ *be regular and nuclear. Let* $T \in N_\lambda(E, F)$ *for any n.l.s.* E *and* F. *Then* $(\alpha_n(T))_n \in \lambda \cdot \lambda^\times$. *Conversely, if* $T \in L(l_2, l_2)$ *and if* $(\alpha_n(T)) \in \lambda \cdot \lambda^\times$ *then* $T \in N_\lambda(l_2, l_2)$.

PROOF. Let $T \in N_\lambda(E, F)$, so T has the representation $Tx = \Sigma \xi_n \langle x, a^n\rangle y^n, \xi \in \lambda \cdot \lambda^\times, (a^n)$ bounded in E' and (y^n) bounded in F. Let $T_N x = \Sigma_{i=1}^N \xi_i \langle x, a^i\rangle y^i$. Then, for all N,

$$\alpha_N(T) \leqslant \|T - T_N\| \leqslant \sup_{\|x\|=1; x\in E} \; \sup_{\|b\|=1; b\in F'} \sum_{i=N+1}^{\infty} |\xi_i| \, |\langle x, a^i\rangle| \, |\langle y^i, b\rangle|$$

$$\leqslant \left(\sup_i \|a^i\|\right)\left(\sup_i \|y^i\|\right) \sum_{i=N+1}^{\infty} |\xi_i|.$$

By Lemma (3.1) and the normality of $\lambda \cdot \lambda^\times$, we see that $(\alpha_n(T)) \in \lambda \cdot \lambda^\times$.

Now consider $T \in L(l_2, l_2)$ with $(\alpha_n(T)) \in \lambda \cdot \lambda^\times$. By the Sprectral Representation Theorem (8.3.1 of [10]), there exist orthonormal systems (u^n) and (v^n) in l_2 such that $Tx = \sum_{n=1}^\infty \alpha_n(T)\langle x, u^n\rangle v^n$, so T is pseudo-$\lambda \cdot \lambda^\times$-nuclear, hence λ-nuclear by (2.1).

(3.6) THEOREM. *Let λ be a regular and nuclear Köthe space. Then $E \in N_\lambda$ if and only if for all barreled-0-neighborhoods U in E there exists a barreled-0-neighborhood V in E such that $(d_n(V, U)) \in \lambda \cdot \lambda^\times$.*

PROOF. First, observe that if $E \in N_\lambda$, then E is nuclear, so that E is a projective limit of Hilbert spaces [10, 4.4.1]. Also for barreled-0-neighborhoods U and V, we have $d_n(V, U) \leqslant \alpha_n(E, (V, U)) \leqslant (n+1)d_n(V, U)$ [10, Proposition 9.1.6]. Thus $(\alpha_n(E(V, U))) \in \lambda \cdot \lambda^\times$ if and only if $(d_n(V, U)) \in \lambda \cdot \lambda^\times$, so applying (3.5) yields the result. □

(3.7) THEOREM (A GROTHENDIECK-PIETSCH CRITERION). *Let λ be a nuclear regular Köthe space. A space $\Lambda(P) \in N_\lambda$ if and only if, for all $a \in P$, there exists $b \in P$ such that $a \prec b$, and an injection $\pi: \mathbf{N} \longrightarrow \mathbf{N}$ such that $\pi(\mathbf{N}) = \{n: a_n \neq 0\}$ and such that $(a_{\pi(n)}/b_{\pi(n)})_n \in \lambda \cdot \lambda^\times$.*

The proof is straightforward from Theorem (3.6). (Cf. [13] for a similar result with $\lambda = \Lambda_\infty(\alpha)$.) □

Given the statement of (3.7), we can now define uniform-λ-nuclearity for any nuclear, regular Köthe space. The original definition was given by Köthe [8] for the space (s) and extended by Dubinsky and Ramanujan [6] to power series space of infinite type. If λ is a nuclear regular Köthe space, then we say that the space $\Lambda(P)$ *is uniformly-λ-nuclear* if and only if there exists a permutation π of $\mathbf{N}$ such that for each $a \in P$ there exists $b \in P$ such that $a \prec b$ and $(a_{\pi(n)}) \in (b_{\pi(n)}) \cdot \lambda \cdot \lambda^\times$. We immediately obtain this theorem.

(3.8) THEOREM. *Let $\lambda = \bigcap (1/a^k) \cdot l_1$ be regular and nuclear and let $\mu = \bigcap (1/b^k) \cdot l_1$ be regular. Then μ is λ-nuclear if and only if μ is uniformly-λ-nuclear.*

PROOF. Let μ be λ-nuclear. By (3.7), for all k there exists m and a permutation π of $\mathbf{N}$ such that $\xi = (b^k_{\pi(n)}/b^m_{\pi(n)}) \in \lambda \cdot \lambda^\times$. By applying Theorem (3.5) to the map $T_\xi: l_1 \longrightarrow l_1$, we see that $\xi \in \lambda \cdot \lambda^\times$ if and only if its decreasing rearrangement $\hat{\xi}$ is in $\lambda \cdot \lambda^\times$. But $\hat{\xi} = b^k/b^m$ by the regularity of (b^k_n). Hence μ is λ-nuclear if and only if for all k there exists m such that $b^k/b^m \in \lambda \cdot \lambda^\times$, so μ is uniformly-λ-nuclear. □

(3.9) COROLLARY. *If* $\lambda = \bigcap (1/a^k) \cdot l_1$ *is regular and nuclear, then* $\lambda \in N_\lambda$ *if and only if* λ *is* (d_2).

PROOF. $\lambda \in N_\lambda$ if and only if for each k there exists m such that $a^k/a^m \in \lambda \cdot \lambda^\times$, which is equivalent to the condition that for each k there exists m and p such that, for all q, $a^k/a^m \cdot a^p/a^q \in l_\infty$, and this is precisely (d_2).

(3.10) COROLLARY. *Let* $\lambda = \bigcap (1/a^k) \cdot l_\infty$ *be regular and nuclear. Then* $\lambda \in N_\lambda$ *if* λ *is* (d_1).

PROOF. Observe that if λ is (d_1) then there exists k_0 such that $a^{k_0} \cdot \lambda$ is (D_1). Thus we may apply Theorem (2.20) of [6] to obtain the result.

4. **Inclusion theorems.** It is of course obvious from Theorem 2.1 that if λ_1 and λ_2 are Köthe spaces for which $\lambda_1 \cdot \lambda_1^\times \subseteq \lambda_2 \cdot \lambda_2^\times$ then $N_{\lambda_1} \subseteq N_{\lambda_2}$. It is not hard to see that the converse fails. However it is possible to give some results in the converse direction, and in special cases to give an equivalent statement. We define the *symmetric hull* of a Köthe space λ by $H(\lambda) = \{t \in \omega$: there is $r \in \mathbf{N}$ and some injection π of $\mathbf{N}$ into $\mathbf{N}$, with $\pi(\mathbf{N}) = \{n: t_n \neq 0\}$, such that $(t^r_{\pi(n)})_n \in \lambda \cdot \lambda^\times\}$.

(4.1) THEOREM. *Let* $N_{\lambda_1} \subseteq N_{\lambda_2}$, *where* λ_1 *and* λ_2 *are Köthe spaces with* λ_2 *regular. Then* $H(\lambda_1) \subseteq H(\lambda_2)$.

PROOF. Let $t \in H(\lambda_1)$, so that there exists $r \in \mathbf{N}$ and π such that $\xi = (t^r_{\pi(n)}) \in \lambda_1 \cdot \lambda_1^\times$. Then $T_\xi \in N_{\lambda_1}(l_1, l_1)$, so if $E = \text{proj lim}_\mathbf{N}\, T_\xi(l_1)$, we have $E \in N_{\lambda_1}$, and thus $E \in N_{\lambda_2}$. Hence there exists $m \in \mathbf{N}$ such that $(T_\xi)^m = T_{(\xi^m)} \in N_{\lambda_2}(l_1, l_1)$. But by Theorem (3.5), there exists an injection σ of $\mathbf{N}$ into $\mathbf{N}$ such that $(\xi^m_{\sigma(n)})_n \in \lambda_2 \cdot \lambda_2^\times$. Thus $t \in H(\lambda_2)$.

(4.2) THEOREM. *Let* λ_1 and λ_2 *be regular spaces which are* d_1, d_2 *or a product of* d_1 *and* d_2 *spaces. The following are equivalent.*

(1) $N_{\lambda_1} \subseteq N_{\lambda_2}$.
(2) $H(\lambda_1) \subseteq H(\lambda_2)$.
(3) $\lambda_1 \cdot \lambda_1^\times \subseteq \lambda_2 \cdot \lambda_2^\times$.

PROOF. We have (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1) so it remains to show that (2) $\Longrightarrow$ (3). For this, observe that if $t \in \lambda_1 \cdot \lambda_1^\times$ then, from (2), $(t^r_{\pi(n)}) \in \lambda_2 \cdot \lambda_2^\times$ for some r and π. But for each $t \in \lambda_1 \cdot \lambda_1^\times$, there exists $\xi \in \lambda_1 \cdot \lambda_1^\times$. with $\xi_1 \geqslant \xi_2 \geqslant \cdots > 0$, and $\xi_n \geqslant |t_n|$. In fact, since $t_n \longrightarrow 0$, we may choose $(n_k) \subseteq \mathbf{N}$ with $n_k < n_{k+1}$ for all k, such that $|t_{n_k}| \geqslant |t_n|$, for $n \geqslant n_{k-1} + 1$, $n_0 = 0$. Let $\xi_n = |t_{n_k}|$ for $n_{k-1} < n \leqslant n_k$. Now $t \in \lambda_1 \cdot \lambda_1^\times$

implies the existence of a^p such that, for all m, $ta^m/a^p \in l_\infty$. Then, for all m,

$$\sup_n \frac{\xi_n a_n^m}{a_n^p} = \sup_k \sup_{n_{k-1}<n\leq n_k} |t_{n_k}| \frac{a_n^m}{a_n^p} = \sup_k |t_{n_k}| \frac{a_{n_k}^m}{a_{n_k}^p} < +\infty,$$

so $\xi \in \lambda_1 \cdot \lambda_1^\times$.

Now we may assume $t_n \downarrow 0$. Using Theorem (3.5) we obtain the fact that $t^r \in \lambda_2 \cdot \lambda_2^\times$. But arguments similar to Lemma (2.10) of [6] and Lemma (3.2) of [14] yield the fact that $t \in \lambda_2 \cdot \lambda_2^\times$, which completes the proof. □

We now consider an arbitrary Köthe space λ and a version of Theorem (3.3). First we must characterize the diagonal maps $T_\xi \in N_\lambda(l_1, l_1)$. A linear map A from l_1 to l_1 and an element $\eta \in \lambda \cdot \lambda^\times$ will be said to form an *admissible pair*, (A, η), *for* λ, if $A \sim (b_i^j y_i^j)$, (b^j) a bounded sequence in l_∞, (y^j) a bounded sequence in l_1, with $\Sigma_j \eta_j b_i^j y_k^j = 0$ when $i \neq k$.

(4.3) Lemma. *$T_\xi \in N_\lambda(l_1, l_1)$ if and only if there exists an admissible pair (A, η) for λ such that $\xi = A\eta$.*

Proof. Suppose that $T_\xi \in N_\lambda(l_1, l_1)$. By Theorem 2, there exist $\eta \in \lambda \cdot \lambda^\times$ and bounded sequences (b^j) and (y^j) in l_∞ and l_1, respectively, such that for all $x \in l_1$

$$T_\xi x = \sum_{j=1}^\infty \eta_j \langle x, b^j \rangle y^j.$$

Let $A \sim (b^j y^j)$. Then, for all i and k, $\delta_{ik}\xi_i = \langle T_\xi e^i, e^k \rangle = \Sigma_{j=1}^\infty \eta_j b_i^j y_k^j$. Hence (A, η) is an admissible pair for λ and $A\eta = \xi$. Conversely, suppose (A, η) is an admissible pair for λ. Then there exist bounded sequences (b^j) and (y^j) in l_∞ and l_1, respectively, such that $A \sim (b_i^j y_i^j)$. Let $\xi = A\eta$, so $\xi \in l_1$. Then $T_\xi \in L(l_1, l_1)$ and for all k and for all $x \in l_1$,

$$\sum_j \eta_j \langle x, b^j \rangle y_k^j = \sum_j \eta_j \left(\sum_{i=1}^\infty x_j b_i^j y_k^j \right)$$

$$= \sum_{i=1}^\infty x_i \sum_j \eta_j b_i^j y_k^j = x_k \xi_k = \langle T_\xi x, e^k \rangle.$$

Hence $T_\xi x = \Sigma_j \eta_j \langle x, b^j \rangle y^j$ for every $x \in l_1$, so that $T_\xi \in N_\lambda(l_1, l_1)$.

(4.4) Theorem. *For any nuclear Köthe space λ and for any sequence space $\Lambda(P)$, $\Lambda(P) \in N_\lambda$ only if $\Lambda(P) \in N_{(s)}$.*

Proof. It suffices to show that if $T_\xi \in N_\lambda(l_1, l_1)$, then $T_\xi \in N_{(s)}(l_1, l_1)$ which by the lemma is equivalent to the existence of an admissible pair (B, γ)

for (s) such that $\xi = B\gamma$. Since T_ξ is in $N_\lambda(l_1, l_1)$, we have an admissible pair (A, η) for λ such that $\xi = A\eta$, where $A \sim (a_i^j y_i^j)$ as in the definition of admissible pair. If $\eta \in \varphi$, then $\eta \in (s)$ so (A, η) is in fact admissible for (s). Thus we may assume that the set $I = \{j \in \mathbf{N}: \eta_j \neq 0\}$ is infinite. Then there exists a bijection of $\mathbf{N}$ onto I such that if $Se^i = e^{\pi(i)}$ for all i, then $S\hat{\eta} = \eta$, where $\hat{\eta}$ is the decreasing rearrangement of η. Let $B = A \cdot S$. Clearly $B\hat{\eta} = \xi$, so we must show that $(B, \hat{\eta})$ is admissible for (s). By Lemma (3.2), $\eta \in \bigcap_{p>0} l_p$, so $\hat{\eta} \in (s)$ (cf. [10, 8.5.5]). Also $Be^j = A(e^{\pi(j)}) = a^{\pi(j)} \cdot y^{\pi(j)}$; so that, for all i and k,

$$\delta_{ik}\xi_i = \sum_{j\in\mathbf{N}} \eta_j a_i^j y_k^j = \sum_{j\in I} \eta_j a_i^j y_k^j = \sum_{n\in\mathbf{N}} \hat{\eta}_n a_i^{\pi(n)} y_k^{\pi(n)}.$$

Hence $(B, \hat{\eta})$ is an admissible pair for λ. □

(4.5) REMARK. Theorem (4.4) says that in particular, any nuclear Fréchet space E with a basis which is λ-nuclear must be (s)-nuclear. Thus the existence of a Fréchet space E which is λ-nuclear but not (s)-nuclear would yield a negative answer to the basis problem in nuclear Fréchet spaces.

5. **Examples.** In this section, we give examples to show the invalidity of some of the earlier results in the case when λ has no regular basis. Where possible we indicate the modification of the theorems which are necessary.

(5.1) In Lemma 3.2, and Theorem 3.3, we observed that $\lambda \cdot \lambda^\times \subseteq \bigcap_{p>0} l_p$ for any nuclear Köthe space λ, and that if λ is regular that $\lambda \cdot \lambda^\times \subseteq (s)$. In general, $\lambda \cdot \lambda^\times$ need not be comparable to (s). It is straightforward to check that

(i) $\lambda \cdot \lambda^\times \subseteq (s)$ if and only if for all k and j there exists m and $M > 0$ with $\overline{\lim}_n\ (a_n^k/a_n^m)^{1/\log n} \leqslant 1/j$, and

(ii) $\lambda \cdot \lambda^\times \supseteq (s)$ if and only if there exists k_0 such that, for all k, $\overline{\lim}_n\ (a_n^k/a_n^{k_0})^{1/\log n} < +\infty$. Let $a^1 = e$ and $a_n^{k+1} = (x_n)^k$, where $x_n = m^2$ if $n = 2^{(m^2)}$ and $x_n = 2^n$ otherwise.

Then $\lambda = \bigcap (1/a^k) \cdot l_1$ has the property that $\lambda \cdot \lambda^\times$ is not comparable to (s). However, from Theorem (4.4), we know that any space $\Lambda(P)$ which is λ-nuclear is also (s)-nuclear.

(5.2) In Theorem (3.5), we proved that if $T \in N_\lambda(E, F)$ with λ regular and nuclear, then $(\alpha_n(T)) \in \lambda \cdot \lambda^\times$. This result is the key to obtaining a Grothendieck-Pietsch criterion. Consider the space $\lambda = \Lambda_1(\alpha) \vee \Lambda_\infty(\alpha), \alpha_n = n$. Then λ has no regular basis [5] and if $\xi_{2n-1} = (½)^n$ and $\xi_{2n} = 0$ for all n, then $\xi \in \lambda \cdot \lambda^\times = \Lambda_1(\alpha)^\times \vee \Lambda_\infty(\alpha)$. But $\hat{\xi} \notin \lambda \cdot \lambda^\times$, so the map $T_\xi: l_1 \longrightarrow l_1$ is λ-nuclear, but $(\alpha_n(T_\xi)) \notin \lambda \cdot \lambda^\times$. However, there is a permutation of $(\alpha_n(T_\xi))$ which is in $\lambda \cdot \lambda^\times$. The existence of this permutation in all cases would allow us to state the Grothendieck-Pietsch criterion in a more general setting than in (3.7). In fact, one can obtain the following.

(5.3) THEOREM. *Let* $\lambda = \lambda_1 \vee \lambda_2, \lambda_1$ *and* λ_2 *Köthe spaces with* λ_2 *regular. Suppose that* $\lambda_1 \cdot \lambda_1^\times \subseteq \lambda_2 \cdot \lambda_2^\times$, *and* λ_2 *is stable* (*i.e.*, $\lambda_2 = \lambda_2 \vee \lambda_2$). *Then* $\Lambda(P) \in N_\lambda$ *if and only if for all* $a \in P$ *there exists* $b \in P$, *with* $a \prec b$, *and a permutation of the nonzero terms of* a/b *which belongs to* $\lambda \cdot \lambda^\times$.

PROOF. We again consider the map $S: l_1 \longrightarrow c_0$ given by $Se^j = \Sigma_{i=1}^{j-1} e^i$. If $\beta \in \lambda \cdot \lambda^\times$, then $\beta \in (\lambda_1 \cdot \lambda_1^\times) \vee (\lambda_2 \cdot \lambda_2^\times) \subseteq \lambda_2 \cdot \lambda_2^\times$. Thus we have $S\beta \in \lambda_2 \cdot \lambda_2^\times$, as was shown in Lemma (3.1). We claim that a permutation of $S\beta \in \lambda \cdot \lambda^\times$. We assume that $\beta_n \geqslant 0$, so that $S\beta$ is decreasing. Now for any decreasing sequence η in $\lambda_2 \cdot \lambda_2^\times$, we can find (n_j) such that $n_j > 2n_{j+1}$ and $\eta_{n_j} < a_j^1/a_j^j$ for all j.

Let $\gamma_{2j-1} = \eta_{n_j}$ and $\gamma_{2j} = \eta_{m_j}$, where m_j is the jth term of $N \backslash \{n_k\}_{k=1}^\infty$. Then $(\gamma_{2j-1})_j < a^1 \cdot (1/a_j^j) \in \lambda_1 \cdot \lambda_1^\times$. Also $m_j \geqslant j$ implies that $\eta_{m_j} < \eta_j$ for all j, so $(\gamma_{2j}) \in \lambda_2 \cdot \lambda_2^\times$. Thus $\gamma \in \lambda \cdot \lambda^\times$, and since γ is a permutation of η we have established the claim.

Now we apply the arguments of (3.5) and (3.7) to finish the proof. □

(5.4) As a consequence of (3.1), we obtain the fact that if λ is a regular and nuclear Köthe space and if $\xi \in \lambda \cdot \lambda^\times$, then there exists a map $T \in N_\lambda(l_1, l_1)$ such that $\alpha_n(T) = \Sigma_{i=n+1}^\infty |\xi_i|$, for all $n = 0, 1, 2, \cdots$. In fact if $\eta_n = \Sigma_{i=n+1} |\xi_i|$ for each n, then $\eta \in \lambda \cdot \lambda^\times$ so that $T_\eta \in N_\lambda(l_1, l_1)$, and $\alpha_n(T_\eta) = \eta_n$, for $n = 1, ,2 \cdots$.

Problem. Does there exist a nuclear Köthe space λ for which the above remark fails?

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